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# Minimax estimation in a deconvolution problem

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**Abstract.** We consider a convolution equation with the right-hand side known with a random noise. *A priori* information that the solution belongs to an ellipsoid in Hilbert space is given. We construct the minimax estimators of the convolution of the solution with a generalized function $h$. As an example the estimation problem of derivative of solution is studied.

## 1. Introduction

Let the convolution equation

$$(a*x)(t)=\int_{-\infty}^{\infty} a(t-s)x(s)\,\mathrm{d}s=f(t) \qquad t\in R^1$$

be given with the kernel $a$ and an unknown right-hand side $f(t)$. Instead of the function $f(t)$ we observe a random process $y(t)=f(t)+\varepsilon\xi(t)$, $\varepsilon>0$. Here $\xi(t)$ is a Gaussian stationary random process with a correlation function $\tau(t)=E\xi(t)\xi(0)$ and $E\xi(t)=0$, $t\in R^1$.

The problem is to estimate a convolution $u(t)=(h*x)(t)$ where $h$ is a given function. We allow the function $h$ to belong to a class of generalized functions. For example, $h$ may be a derivative, $(h*x)(t)=\mathrm{d}x(t)/\mathrm{d}t$.

In this paper we develop the minimax approach to this problem similar to [1, 2]. This approach may be described as follows.

Let *a priori* information about the solution be given

$$x(t)\in Q_T=\left\{x:x(t)=\int_{-\infty}^{\infty} b(t-s)z(s)\,\mathrm{d}s,\ \int_{-\infty}^{\infty} z^2(s)\,\mathrm{d}s<2P_0T,\ \mathrm{supp}\, z(s)\subseteq(-T,T)\right\} \qquad (1.1)$$

where $P_0>0$ and $b\in L_2(R^1)$.

The minimax risk of any estimator $u^*(t)=u^*(t,y)$ equals

$$\rho_T(u^*)=\sup_{x\in Q_T}(2T)^{-1}\int_{-T}^{T} E_x(u^*(t)-u(t))^2\,\mathrm{d}t.$$

We need to construct the minimax families of estimators $u^{**}=\{u_T^{**}(t)\}_0^\infty$ such that

$$\rho_T(u_T^{**})=\inf_{u^*\in\Omega}\rho_T(u^*)(1+\mathrm{O}(1)) \qquad (1.2)$$

as $T\to\infty$. Here $\Omega$ is the class of all estimators.

We call

$$\Psi_\varepsilon = \limsup_{T\to\infty} \rho_T(u_T^{**})$$

the optimal minimax risk in this problem.

A similar approach is also developed for the scheme with discrete time. Assume we observe the random variables $y(t_j) = y(j\Delta), j = 0, \pm 1, \pm 2, \ldots, \Delta > 0$. The problem is to construct minimax families of estimators $u^{**} = \{u_{T\Delta}^{**}(y)\}_0^\infty$ such that

$$\limsup_{\Delta\to 0} \limsup_{T\to\infty} \rho_T(u_{T\Delta}^{**})$$

$$= \limsup_{\Delta\to 0} \limsup_{T\to\infty} \inf_{u^*\in\Omega} \rho_T(u^*)(1+\mathrm{o}(1)). \tag{1.3}$$

Denote by $\Lambda_\varepsilon$ the left-hand side of (1.3). For any function $z \in L_2(r^1)$ denote

$$Z(\omega) = \int_{-\infty}^{\infty} \exp\{2\pi i\omega t\} z(t)\,\mathrm{d}t$$

$$\|z\| = \left(\int_{-\infty}^{\infty} |z(t)|^2\,\mathrm{d}t\right)^{1/2}.$$

Using this notation we may write

$$Q_T = \{x\colon X(\omega) = B(\omega)Z(\omega), \|z\|^2 < 2P_0T, \operatorname{supp} z \subseteq (-T, T)\}.$$

In the case $|B(\omega)|^2 = (1+|\omega|^{2\beta})^{-1}, \beta > 0$, such an assignment resembles that of the ball in the Sobolev space $W_2^{2\beta}((-T, T))$. A comprehensive discussion of this definition of *a priori* information is given in [2].

For the estimation problem of the solution $x(t)$ similar results have been obtained in our papers [1, 2]. In those papers we considered the classes $\Omega$ of all estimators and linear ones respectively. Similar approaches of estimation $h * x$ have been developed in [3–5]. The investigations of these papers are concerned with robustness problems. The minimax estimators constructed in our paper are also robust.

The proof of this paper's results is based on the following arguments. We define a linear estimator on which the asymptotical bound $\Lambda_\varepsilon$ of minimax risks is achieved. After that we construct a family of Bayes estimators with the Bayes risks tending to $\Lambda_\varepsilon$ as $T\to\infty$. Hence using the argument that the Bayes risk does not exceed the minimax one we obtain the paper's results.

*A priori* Bayes measures of Bayes estimators are conditional measures of Gaussian random processes $\zeta_T(s), s \in R^1$, under condition $\zeta_T \in Q_T$. The introduction of such a condition is caused by *a priori* information $x(t) \in Q_T$. The random processes $\zeta_T$ are defined as follows. At first we introduce a Gaussian stationary random process $e(t), t \in R^1$. The spectral density of this process is found as a solution of some extremal problem maximizing Bayes risk. Then we put $\zeta_T(s) = (b * e_T)(s), s \in r^1$, where $e_T(t) = e(t)$ for $|t| < T$ and $e_T(t) = 0$ for $|t| > T$.

Similarly to lemma 4 in [6] (see also [7]) it is shown that $P(\zeta_T(s) \in Q_T) = 1 + \mathrm{o}(1)$ as $T\to\infty$ and the effects caused by the events $e_T \notin Q$ can be neglected. The proof of these assertions is based on the law of large numbers. It has a standard structure and is omitted. Thus the calculation problem of the asymptotic of Bayes risk with conditional *a priori* measures $P_T$ is replaced by the same one having unconditional *a priori* measures generated by a random process $\zeta_T$. In lemma 2 we show that the Bayes risks in the last problem converge as $T\to\infty$ to the risk of Wiener filtration. The Wiener filtration is based on *a priori* information that the solution $x$ is a realization of the stationary random process $\zeta(s)=(b*e)(s)$, $s\in R^1$.

## 2. Main results

In this section the main theorems will be formulated and the examples of asymptotics of minimax risks as $\varepsilon\to 0$ will be indicated. The proof of theorems will be given in section 3.

For any $s\in R^1$ denote $(s)_+=\max\{s,0\}$. Introduce the function

$$\Phi(\mu)=\varepsilon^2\int_{-\infty}^{\infty}|A(\omega)B(\omega)|^2(|H(\omega)B(\omega)|\mu^{-1}-1)_+R(\omega)\,d\omega. \tag{2.1}$$

Denote $\mu_\varepsilon=\sup\{\mu\colon \Phi(\mu)\geq P_0,\ \mu>0\}$. Make the following assumptions:

(i) The functions $a(t)$, $b(t)$, $r(t)$, $|H(\omega)B(\omega)|$ belong to $L_2(R^1)$ and $L_1(R^1)$ simultaneously. The functions $|A(\omega)B(\omega)|^2R^{-1}(\omega)$ and $|H(\omega)B(\omega)|$ are bounded.

(ii) $|A(\omega)B(\omega)|>0$ for all $\omega\in R^1$ and $H(\omega)B(\omega)\to 0$ as $|\omega|\to\infty$.

(iii) There exists a decreasing function $g(\tau)\in L_1(R^1_+)$ such that $|A(\omega)B(\omega)|^2<g(|\omega|)$ for all $\omega\in R^1$.

For all $\mu>0$ define the kernel $k_\mu(t)$ by its Fourier transformation $K_\mu(\omega)=A^{-1}(\omega)H(\omega)(1-\mu|B(\omega)H(\omega)|^{-1})_+$.

*Theorem 1.* Assume (i) and (ii). Then the family of estimators $u_T^{**}(t,y)=(k_\mu*y)(t)$ with the value of parameter $\mu=\mu_\varepsilon$ is asymptotically minimax. The minimax risk has the following asymptotics:

$$\Psi_\varepsilon=\varepsilon^2\int_{-\infty}^{\infty}|A^{-2}(\omega)H^2(\omega)|(1-\mu_\varepsilon|B(\omega)H(\omega)|^{-1})_+R(\omega)\,d\omega. \tag{2.2}$$

Note that the estimators $u_T^{**}$ do not depend on $T$.

*Theorem 2.* Let the random process $y(t)$ be observed in discrete points $t_j=j\Delta$, $j=0,\pm1,\pm2,\ldots$ Let assumptions (i)-(iii) be satisfied. Then the family of estimators

$$u_{T\Delta}^{**}(t,y)=\Delta\sum_{j=-\infty}^{\infty}k_\mu(t-t_j)y(t_j)$$

is asymptotically minimax. The asymptotic of minimax risk $\Lambda_\varepsilon$ equals to the right-hand side of (2.2).

*Remark.* The assumption of boundedness of $|A(\omega)B(\omega)|^2R^{-1}(\omega)$ seems to be unnatural. Note that if this assumption is not satisfied and $|B(\omega)H(\omega)|^2|\omega|^{1+\delta}\to 0$ as $|\omega|\to\infty$ for some $\delta>0$ then the estimation problem is not ill-posed. For this case the order of $\Lambda_\varepsilon$ is $\varepsilon^2$.

*Remark.* The results of theorem 1 also hold if $\xi(t)$ is a white noise. In this case $R(\omega)=1$ for all $\omega\in R^1$ and assumptions $r(t)\in L_1(R^1)$ are omitted. For theorem 2 a similar assertion is also valid. In this version of theorem 2 $\xi(t_j)$ are independent Gaussian random variables. These results are obtained by slight modification of the proof of theorems 1 and 2.

*Remark.* In practical problems the observations $\xi(t_j)$ often only depend on near observations $\xi(t_{j_1})$. In order to involve such a situation we considered in [1] the model of observations $\xi(t_j)$ with the correlation function $r(t/\alpha_\Delta)$ where $\alpha_\Delta \to 0$ as $\Delta \to 0$. Similar results can also be proved for this problem.

*Remark.* As we said, $h$ may be a generalized function. For example, if $u(t) = \mathrm{d}x(t)/\mathrm{d}t$ then $H(\omega) = 2\pi i\omega \exp\{2\pi i t\omega\}$. If $a(t)$ is a delta function we obtain the usual problem of derivative estimation (see [8, 9]).

Now we give examples of asymptotics $\Psi_\varepsilon$ as $\varepsilon \to 0$.

*Example 1.* Let $2\zeta - 2\beta + 1 < 0$, $2\gamma - 2\zeta - 2\tau + 1 > 0$ and

$$|B(\omega)||\omega|^\beta = W(1 + o(1)) \tag{2.3}$$

$$|A(\omega)||\omega|^\gamma = U(1 + o(1)) \tag{2.4}$$

$$R(\omega)|\omega|^{2\tau} = G(1 + o(1)) \tag{2.5}$$

$$|H(\omega)||\omega|^\zeta = N(1 + o(1)) \tag{2.6}$$

as $|\omega| \to \infty$. Then

$$\Psi_\varepsilon = \varepsilon^{4\sigma} N^2 U^{-2} G C_1^{1-2\sigma} (\beta + \zeta)/((2\gamma + \beta - 2\tau - \zeta + 1)(\gamma + \zeta - \tau + 0.5))$$

where

$$C_1 = P_0 G^{-1} W^2 U^2 (2\sigma)^{-1} (2\gamma + \beta - 2\tau - \zeta + 1)$$

$$\sigma = (\beta + \zeta)/(2\gamma + 2\beta - 2\tau + 1).$$

*Example 2.* Assume (2.4)–(2.6) and

$$|A(\omega)||\omega|^\rho \exp\{d|\omega|^\gamma\} = U(1 + o(1)) \tag{2.7}$$

as $\omega \to \infty$. Assume also $2\zeta + 2\beta > 0$. Then

$$\Psi_\varepsilon = P_0 W^2 N^2 |\ln \varepsilon / d|^{-2(\zeta + \beta)/\gamma}.$$

*Example 3.* Assume (2.4), (2.5) and (2.7) and

$$|H(\omega)||\omega|^\alpha \exp\{p|\omega|^\zeta\} = N(1 + o(1))$$

as $\omega \to \infty$. Assume also $0 < \zeta < \gamma$. Then

$$\Psi_\varepsilon = P_0 W^2 N^2 |\ln \varepsilon / d|^{-(2\beta + 2\alpha)/\gamma} \exp\{-2p|\ln \varepsilon / d|^{\zeta/\gamma}\}.$$

## 3. Proof of theorems

The proof of theorem 1 is based on lemmas 1 and 2.

For any function $G(\omega)$ denote

ess sup $G(\omega) = \inf\{c$: Zebesque measure $\{\omega: G(\omega) > c, \omega \in R^1\}$ equals zero$\}$.

*Lemma 1.* Assume (i) and (ii). Then for any function $k \in L_2(R^1)$

$$\Psi_\varepsilon \leqslant \varepsilon^2 \int_{-\infty}^{\infty} |K(\omega)|^2 R(\omega) \,\mathrm{d}\omega + P_0 \operatorname{ess\,sup} |B(\omega)|^2 |H(\omega) - K(\omega) A(\omega)|^2. \tag{3.1}$$

The proofs of similar results are given in [2, 7]. We omit these arguments here. Note that the right-hand side of (3.1) equals the limit of $\rho_T(k * y)$ as $T \to \infty$.

Put $K(\omega) = K_{\mu_\varepsilon}(\omega)$, $\omega \in R^1$. Then arguing similarly to [2] and [6] we obtain that $\Psi_\varepsilon$ does not exceed the right-hand side of (2.2).

*Lemma 2.* Assume (i) and (ii). Let $v \in L_2(R^1)$ and $\|V\|^2 < P_0$, ess sup$|V(\omega)| < \infty$. Then

$$\Psi_\varepsilon \geq \varepsilon^2 \int_{-\infty}^{\infty} \frac{R(\omega)|H(\omega)B(\omega)V(\omega)|^2}{|A(\omega)B(\omega)V(\omega)|^2 + \varepsilon^2 R(\omega)} \, d\omega. \tag{3.2}$$

Note that the right-hand side of (3.2) is the risk of Wiener filtration with *a priori* information that a solution $x(t)$ is a realization of a Gaussian stationary random process with spectral density $|V(\omega)B(\omega)|^2$.

It is easy to see that the supremum of the right-hand side of (3.2) is achieved on the function $|V(\omega)|^2 = \varepsilon^2 R(\omega) \times |A(\omega)B(\omega)|^{-2}(\mu_\varepsilon^{-1}|H(\omega)B(\omega)| - 1)_+$. This supremum equals $\Psi_\varepsilon$. Thus theorem 1 follows from lemmas 1 and 2.

*Proof of lemma 2.* Let $e(t)$, $t \in R^1$, be a Gaussian stationary random process with spectral density $|V(\omega)|^2$. Define the random process $e_T$ as we said in the introduction. We prove only that the Bayes risks of Bayes estimators having *a priori* measures of random processes $\zeta_T(t) = (b * e_T)(t)$ converge to the corresponding risk of Wiener filtration. The other arguments are obtained by easy modification of those developed in [1, 6, 7] and are omitted.

By virtue of the *a priori* measures being Gaussian the Bayes estimators equal

$$\hat{u}_T(t, y) = \int_{-\infty}^{\infty} k_T(t, s) y(s) \, ds. \tag{3.3}$$

Here the kernel $k_T(t, s)$ satisfies the equation

$$Eu_T(t)f(s) = E \int_{-\infty}^{\infty} k_T(t, w) f(w) \, dw f(s) + \varepsilon^2 \int_{-\infty}^{\infty} k_T(t, w) r(w - s) \, dw \tag{3.4}$$

for all $t \in (-T, T)$ and $s \in R^1$.

If we make the formal limit transition on $T \to \infty$ we obtain the similar equation for the estimator of Wiener filtration

$$\hat{u}(t, y) = \int_{-\infty}^{\infty} k(t - s) y_p(s) \, ds \tag{3.5}$$

$$E(h * \zeta)(t) f_p(s) = E(k * f_p)(t) f_p(s) + \varepsilon^2 (k * r)(t - s) \tag{3.6}$$

where

$$\zeta(t) = (b * e)(t) \qquad f_p(t) = (a * \zeta)(t) \qquad y_p(t) = f_p(t) + \varepsilon \xi(t)$$
$$t \in R^1 \qquad s \in R^1.$$

Applying the Fourier transformation to (3.6) we obtain

$$K(\omega) = H(\omega) A(-\omega) |B(\omega) V(\omega)|^2 (|A(\omega) B(\omega) V(\omega)|^2 + \varepsilon^2 R(\omega))^{-1}.$$

For the proof of lemma 2 it remains to show

$$\lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} EE_e(\hat{u}_T(t) - \hat{u}(t))^2 \, dt = 0 \tag{3.7}$$

where the first mathematical expectation $E$ is taken over the random process $e(t)$, $t \in R^1$.

Denote by $K_T(t, \omega)$ the Fourier transformation of $k_T(t, s)$ with respect to the second variable $s$.

Put $S(T) = T - d(T)$ where $d(T) \to \infty$, $d(T)/T \to 0$ as $T \to \infty$.

In order to prove (3.7) we now show

$$\|K_T(t, \omega) - K(\omega)\exp\{2\pi i\omega t\})R^{1/2}(\omega)\| = o(1) \tag{3.8}$$

uniformly in $t \in (-S(T), S(T))$ as $T \to \infty$.

Introduce the function $D(\omega) = \sin(2\pi T\omega)/(2\pi T\omega)$. For any function $q(s)$ and $t \in R^1$ denote $q_t(s) = q(t+s)$. Put $v_2(t) = (v * v)(t)$. For all $t \in R^1, T > 0$, define the projector $P_{Tt}$ such that $P_{Tt}q(s) = q(s)\chi_{(-T-t, T-t)}(s)$ for any function $q \in L_2(R^1)$. Here $\chi_{(-T-t,T-t)}(s)$ is the indicator of interval $(-T-t, T-t)$. Put $P_T = P_{T0}$. Denote by $I$ the identity operator.

Define the functions $G_t(\tau) = A(\tau)H(-\tau)|B(\tau)V(\tau)|^2 \exp\{2\pi i t\tau\}$ and $G_{Tt}(\tau) = A(\tau)B(\tau)Y_{Tt}(\tau)$ where

$$Y_{Tt}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\tau-\omega)|V(\omega)|^2 D(\omega-\omega_1)H(-\omega_1)B(-\omega_1)$$
$$\times \exp\{-2\pi i t\omega\}\, d\omega\, d\omega_1.$$

Applying the Fourier transformation to (3.4) we obtain

$$G_{Tt}(\tau) = \int_{-\infty}^{\infty} K_T(t, -\tau_1)\Gamma_T(\tau, \tau_1)\, d\tau_1 + \varepsilon^2 R(\tau)K_T(t, \tau) \tag{3.9}$$

where

$$\Gamma_T(\tau, \tau_1) = A(\tau)B(\tau)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\tau-\omega)|V(\omega-\omega_1)|^2 D(\omega_1-\tau_1)\, d\omega\, d\omega_1$$
$$\times B(-\tau_1)A(-\tau_1).$$

Further, by $C$ we shall denote arbitrary constants.

For all $t \in (-S(T), S(T))$, using boundedness of $A(\tau)B(\tau)R^{-1/2}(\tau)$ we have

$$\begin{aligned}
&\|R^{-1/2}(G_{Tt} - G_t)\| \\
&\quad \leqslant C\|Y_{Tt}(\tau) - |V(\tau)|^2 H(-\tau)B(-\tau)\exp\{2\pi i t\tau\}\| \\
&\quad \leqslant C\|P_T v_2 * P_T(h * b)_t - v_2 * h * b_t\| \\
&\quad < C\|P_T v_2 * (I - P_T)(h * b)_t\| + C\|(I - P_T)(v_2 * h * b_t)\| \\
&\quad < C\|v_2 * (I - P_{Tt})(h * b)\| + C\|(I - P_{Tt})(v_2 * h * b)\| \\
&\quad < C \operatorname{ess\,sup}|V(\omega)|^2 \|(I - P_{d(t)})(h * b)\| \\
&\qquad + C\|(I - P_{d(T)})(v_2 * h * b)\| = o(1)
\end{aligned} \tag{3.10}$$

as $T \to \infty$.

Let $\Gamma$ be the multiplication operator in $|A(\omega)B(\omega)V(\omega)|^2$. It is easy to see that

$$R^{1/2}(\omega)K(\omega)\exp\{2\pi i t\omega\} = (I + R^{-1/2}\Gamma R^{-1/2})^{-1}R^{-1/2}(\omega)G_t(\omega).$$

The operator $\Gamma_T$ is self-conjugated and non-negative. Therefore we may write (3.9) as follows:

$$R^{1/2}(\omega)K_T(t,\omega)=(I+R^{-1/2}\Gamma_T R^{-1/2})^{-1}R^{-1/2}(\omega)G_{Tt}(\omega).$$

The function $A(\omega)B(\omega)R^{-1/2}(\omega)$ is bounded. Hence, using boundedness of operators $D_\tau * |V|^2 * D_{-\tau_1}$; we obtain that the operators $R^{-1/2}\Gamma_T R^{-1/2}$ are uniformly bounded in $T>0$. Therefore for any $\delta>0$ there exists such a polynom $\Pi$ that $L_2$-norms

$$(I+R^{-1/2}\Gamma_T R^{-1/2})-\Pi(R^{-1/2}\Gamma_T R^{-1/2})$$

$$(I+R^{-1/2}\Gamma R^{-1/2})^{-1}-\Pi(R^{-1/2}\Gamma R^{-1/2})$$

do not exceed $\delta$. Hence, by (3.10), we obtain that (3.9) holds if $\|(R^{-1/2}(\Gamma_T-\Gamma)R^{-1/2})F_t\|=o(1)$ uniformly in $t\in(-S(T),S(T))$ as $T\to\infty$ for any $F_t=(R^{-1/2}\Gamma R^{-1/2})^m G_t$, $m=0,1,\ldots$.

Denote $Z_t(\omega)=R^{-1/2}(\omega)A(\omega)B(\omega)F_t(\omega)$. We have

$$\|R^{-1/2}(\Gamma_T-\Gamma)R^{-1/2}F_T\|<C\|P_T v_2 * P_T z_t - v_2 * z_t\|=o(1) \tag{3.11}$$

uniformly in $t\in(-S(T),S(T))$ as $T\to\infty$. The equality in (3.11) is proved similar to (3.10).

We have

$$E\left(\int_{-\infty}^{\infty}(k_T(t,s)-k(t-s))f(s)\,ds\right)^2$$
$$\leqslant \operatorname{ess\,sup}|V(\omega)|^2\|P_T(k_T(t,s)-k(t-s)) * (a * b)(s)\|^2$$
$$\leqslant C\|(K_T(t,\omega)-K(\omega)\exp\{2\pi i\omega t\})R^{1/2}(\omega)\|^2=o(1) \tag{3.12}$$

uniformly in $t\in(-S(T),S(T))$ as $T\to\infty$.

We have also

$$E\int_{-S(T)}^{S(T)}dt\int_{-\infty}^{\infty}|k_T(t,s)\xi(s)\,ds-(k * \xi)(t)|^2\,dt$$
$$=\int_{-S(T)}^{S(T)}dt\int_{-\infty}^{\infty}R(\omega)|K_T(t,\omega)-K(\omega)\exp\{2\pi i\omega t\}|^2\,d\omega=o(T) \tag{3.13}$$

as $T\to\infty$.

Now (3.12) and (3.13) imply (3.7). This completes the proof of lemma 2. □

*Remark.* Assume $b$ is a delta function. Then *a priori* measures are generated by Gaussian stationary random processes $e_T(s)$, $s\in(-T,T)$. We proved that Bayes risks for these *a priori* measures tend to the risk of corresponding Wiener filtration and Bayes kernels also tend to the Wiener filter as $T\to\infty$. In such a way we obtain the modification of the problem of Wiener filtration. For the estimation problem of $x(t)$ similar results have been formulated in our paper [1].

The proof of theorem 2 is based on lemmas 3 and 4. These lemmas are the analogues of lemmas 1 and 2 respectively.

*Lemma 3.* Assume (i)-(iii). Then for any function $K \in L_2(r^1)$ it holds

$$\Lambda_\varepsilon \leqslant \Delta^{-2} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |K(\omega+j/\Delta)|^2 R(\omega)\,d\omega$$
$$+ P_0 \Bigg\{ \text{ess sup} |B(\omega)|^2 |H(\omega) - \Delta^{-1} A(\omega) K(\omega)|^2$$
$$+ \Bigg[ \text{ess sup}\, \Delta^{-2} \sum_{l \neq 0} \sum_{j \neq 0} |K(\omega) - 1/\Delta(\omega + (j-1)/\Delta)$$
$$\times B(\omega + (j-1)/\Delta)|^2 \Bigg]^{1/2} \Bigg\}^2. \tag{3.14}$$

The proof is similar to that of lemma 2 in [7] and is omitted.

*Lemma 4.* Assume (i)-(iii). Let $v \in L_2(R^1)$, $\|v\|^2 < P_0$ and ess sup$|V(\omega)| < \infty$. Then

$$\Lambda_\varepsilon \geqslant \int_{-\infty}^{\infty} \Bigg\{ |H(\omega)B(\omega)V(\omega)|^2 - |H(\omega)(B(\omega)V(\omega))^2|^2$$
$$\times \left( \sum_{j=-\infty}^{\infty} |A(\omega+j/\Delta)B(\omega+j/\Delta)V(\omega+j/\Delta)|^2 + R(\omega+j/\Delta)) \right)^{-1} \Bigg\} d\omega \tag{3.15}$$

for all $0 < \Delta < \Delta_0$.

Let the functions $K$ and $|V|^2$ be the same as in the proof of theorem 1. Then theorem 2 follows directly from (3.14) and (3.15).

*Proof of lemma 4.* The proof is similar to that of lemma 2. It is based on the same definition of the Gaussian stationary random process $e(s)$ and calculation of asymptotics of Bayes risks as $T \to \infty$. The difference in analytical methods is insignificant. All such differences we may clearly see in the proof of the analogue of (3.11). The arguments will be given only for this proof.

The discrete Fourier transformation of $r(t_j)$ is

$$R_1(\tau) = \sum_{j=-\infty}^{\infty} R(\tau + j/\Delta) > C > 0$$

where $\tau \in (-(2\Delta)^{-1}, (2\Delta)^{-1})$.

The kernel of operator $\Gamma_T(\tau, \tau_1)$ equals

$$\sum_{j=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A(\tau - j/\Delta)B(\tau+j/\Delta) \int_{-\infty}^{\infty} D(\tau + j/\Delta - \omega)|V(\omega)|^2$$
$$\times D(\omega - \tau_1 - j_1/\Delta)\,d\omega\, B(-\tau_1 - j_1/\Delta)A(-\tau_1 - j_1/\Delta).$$

Let $\|\cdot\|_C$ be $L_2((-(2\Delta)^{-1}, (2\Delta)^{-1}))$-norm. Let $F$ belong to $L_2((-(2\Delta)^{-1}, (2\Delta)^{-1}))$. Define the function $M_t(\tau)$, $\tau \in R^1$, as follows. Put

$$M_t(\tau) = A(-\tau)B(-\tau)F(\tau - j/\Delta)\exp\{2\pi i \omega t\}$$

for

$$\tau \in ((2j-1)/(2\Delta), (2j+1)/(2\Delta)), j = 0, \pm 1, \pm 2, \ldots.$$

It is easy to see that $M_t(\tau) \in L_2(R^1)$.

Introduce the multiplication operator $\Gamma$ by

$$\sum_{j=-\infty}^{\infty} |A(\tau+j/\Delta)B(\tau+j/\Delta)V(\tau+j/\Delta)|^2.$$

Similarly to (3.10) it suffices to estimate

$$\begin{aligned}\|(\Gamma_T-\Gamma)F_t\| &= \int_{-(2\Delta)^{-1}}^{(2\Delta)^{-1}} \Bigg| \sum_{j=-\infty}^{\infty} A(\tau+j/\Delta)B(\tau+j/\Delta) \\ &\quad \times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(\tau+j/\Delta-\omega)|V(\omega)|^2 D(\omega-\tau_1)M_t(\tau_1)\,\mathrm{d}\omega\,\mathrm{d}\tau_1 \right. \\ &\quad \left. -|V(\tau+j/\Delta)|^2 M_t(\tau) \right) \Bigg|^2 \mathrm{d}\tau. \end{aligned} \tag{3.16}$$

It follows from assumption (ii) that

$$\sum_{j=-\infty}^{\infty} |A(\tau+j/\Delta)B(\tau+j/\Delta)|^2 < C < \infty \qquad \tau\in R^1.$$

Hence the right-hand side of (3.16) does not exceed

$$\begin{aligned} &\int_{-(2\Delta)^{-1}}^{-(2\Delta)^{-1}} \sum_{j=-\infty}^{\infty} \Bigg| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\tau+j/\Delta-\omega)|V(\omega)|^2 D(\omega-\tau_1) \\ &\qquad \times M_t(\tau_1)\,\mathrm{d}\omega\,\mathrm{d}\tau_1 - |V(\tau+j/\Delta)|^2 M_t(\tau)\Bigg|^2 \mathrm{d}\tau \\ &\quad = \|P_T v_2 * P_T m_t - v_2 * m_t\|^2 = \mathrm{o}(1) \end{aligned}$$

as $T\to\infty$.

This completes the proof of the analogues of (3.11) and lemma 4. □

## 4. Discussion

The model is intended for the estimation of the convolution of the solution $x(t)$ with a given smooth or generalized function $h$. The use of the smooth function $h$ for the estimation is natural for the study of the behaviour of irregular solutions. The estimation problem of the convolution of a solution with a generalized function often arises in practice as the estimation problem of the solution derivative.

In our papers [1, 2] we indicated a whole range of merits of the model and estimators arising. The assignment of the sets $Q_T$ is natural. The minimax estimators $u^{**}$ are robust and depend on the assignment of noise only through the parameter of regularization $\mu$. The Fourier transformation of estimators kernel has a finite support.

The deficiencies of the model are as follows. We ought to have introduced the limit transition as $T\to\infty$. As a consequence the signal $x(t)$ has the support expanding as $T\to\infty$ and the powers of signal and noise tend to infinity as $T\to\infty$. The ratio of powers of signal and noise naturally remained a constant. It is very difficult to do without such an assumption. The model of Wiener filtration also contained assumptions of such a type.

The lower bound of minimax risk is achieved on fast oscillating functions with the period of oscillation tending to zero as $\varepsilon \to 0$. It is clear that the real signal $x$ may not be fast oscillating. Then the risk will have another order of convergence as $\varepsilon \to 0$. This is the usual situation for asymptotic minimax problems. Our results are obtained for fixed $\varepsilon > 0$.

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